Simplified Football: A Two-Stage Game with a Mixed-Strategy Equilibrium
(by Avinash Dixit and Susan Skeath)

In football, when the offense needs to gain 20 yards and has two plays to do it, commentators often discuss whether it is better to try for the whole lot in one play, or go for half the distance each time. We show you how to think through this, and how to startle your friends with your insight.

Suppose the offense has two more plays left, labeled the third down and the fourth down, to gain 20 yards, in which case it wins; otherwise it loses. The offense's coach has two plays for this situation, one that will cover 10 yards if successful, the other, 20. The opposing coach, knowing this, can set his defense to cover either. The offense's probabilities of success for the possible combinations of the two sides' choices in any single down are shown in Figure 1. (This is not the payoff matrix of the game itself.) The payoffs for the ultimate outcomes are as follows: if the offense wins it gets 1, if it loses it gets 0. This is a zero-sum game.

<u>Figure 1 - Success probability table in two-play football example</u>

		Defense	
		10	20
Offense	10	4/5	1
	20	1	1/2

As usual, we solve this game backward. Begin by considering the fourth down. This may occur with either 10 or 20 yards to go depending on what happened on the earlier third down. We have to consider these two cases separately.

First we look at the case where 20 yards are needed at the 4th down. Here the offense loses the game even if it successfully completes the 10 yard play. Therefore it is forced to use the 20 yard play. The defense, knowing this, uses the 20 yard defense. The payoff to the offense is 1/2. Technically, the payoff table for the offense is as shown in Figure 2. This game is dominance solvable, and the equilibrium is (20,20) leading to an expected payoff of 1/2 for the offense, as we just argued.

Figure 2 - Payoff table for fourth down, 20 yards to go

		Defense	
		10	20
Offense	10	0	0
	20	1	1/2

Next consider the case where only 10 yards are needed on the 4th down. Now either successful play wins the game for the offense. Therefore its payoff equals its probability of success. The payoff table is shown in Figure 3.

Figure 3 - Payoff table for fourth down, 10 yards to go

		Defense	
		10	20
Offense	10	4/5	1
	20	1	1/2

In this payoff table there is no dominance for either pure strategy, nor a Nash equilibrium in pure strategies, so we look for a mixed strategy Nash equilibrium. If the offense mixes in proportions of p for the 10-yard play and (1-p) for the 20-yard play, the condition for the defense to be indifferent between its two plays is

$$0.8\,p + (1-p) = p + 0.5\,(1-p), \text{ or } 1 - 0.2\,p = 0.5 + 0.5\,p, \text{ or } 0.7\,p = 0.5,$$

which yields p = 5/7. Similarly the defense's equilibrium mix has the probability q = 5/7 of choosing to cover the 10-yard play. The resulting expected payoff to the offense can be found by evaluating the result of the offense's p-mix when played against either of the defense's pure strategies, say its 10-yard play. This gives 1 - (5/7)/5 = 1 - 1/7 = 6/7.

A more complicated way to get the same payoff is to calculate the average of all four possible strategy combinations using the right combined probabilities of their occurrence when both players mix: (5/7)(5/7)(4/5) + (5/7)(2/7)1 + (2/7)(5/7)1 + (2/7)(2/7)(1/2) = 42/49 = 6/7.

Now consider the third down. Here we need to trace the consequences of each pair of choices. For example, if each side uses the 10-yard move, then (i) with probability 4/5 the offense succeeds, gains 10 yards, and then on the 4th down with 10 to go it has the expected payoff of 6/7 calculated above, and (ii) with probability 1/5 the offense fails, is left with 4th down and 20, and has the expected payoff of _ calculated above.

Therefore (10,10) yields the expected payoff (4/5)(6/7) + (1/5)(1/2) = 55/70 = 11/14 to the offense.

Similarly (10,20) yields 1 (6/7) = 6/7, (20,10) yields 1, and (20,20) yields (1/2) 1 + (1/2)(1/2) = 3/4. Using all these calculations, we have the payoff table for the third down, shown in Figure 4.

Figure 4 - Payoff table for third down

		Defense	
		10	20
Offense	10	11/14	6/7
	20	1	3/4

This table shows no dominance and we look for a mixed-strategy equilibrium. We find the offense's p-mix by the usual condition:

$$(11/14)\,p + (1-p) = (6/7)\,p + (3/4)\,(1-p), \text{ or } 1 - 3p/14 = 3/4 + 3p/28,$$

which yields $1/4 = 9p/28$, or $p = 7/9$. Similarly, for the defense's mixture,

$(12-q)/14 = (3+q)/4$ yields $3/28 = 18\,q/56$, or $q = 1/3$

The expected payoff to the offense is 5/6.

If the numbers in the initially specified table of success probabilities were different, these mixture probabilities and payoffs would also be different. But three general points stand out from the analysis. First, neither of the sports commentators' doctrines, "take two stabs at 20" nor "go for half the distance each time" is appropriate as a pure strategy; mixing remains useful for the usual reason in zero-sum games, namely that any other consistent pattern would be exploited by the opponent.

Second, in the 3rd down mix, the offense's probability of choosing the 10 yard play is quite high. There are two different ways of looking at this: (i) The 10 yard play is safer – it is the percentage play in the sense that we saw in Chapter 5; its payoffs against the defense's choices are closer together than those of the 20 yard play: 6/7 - 11/14 = 1/14 while 1 - 3/4 = 1/4. (ii) The defense covers the 20 yard play 2/3 of the time because losing it is very costly to the defense. Therefore the offense uses the less covered 10-yard play more often.

And, third, with two plays, an added point in favor of the 10 yard play on the 3rd down is that if it succeeds, then you get the advantage of being able to mix and keep the defense guessing on the 4th down. If you try the 20 yard play on 3rd down and fail, then you are forced to go for 20 on the 4th and the defense can cover this. Hence the greater weight on the 10 yard play on 3rd down as compared to 4th and 10: 7/9 > 5/7.