Stripped-Down Poker:
A Classroom Game with Signaling and Bluffing

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Abstract

We present a simplified, "stripped down" version of poker as an instructional classroom game. While Stripped-Down Poker is extremely simple, it nevertheless provides an excellent illustration of a number of topics: signaling, bluffing, mixed strategies, the value of information, and Bayes' Rule. We begin with a description of Stripped-Down Poker: how to play it, what makes it an interesting classroom game, and how to teach its solution to students. We describe how signaling, bluffing, etc, emerge naturally as important features of the game. Then we discuss possible applications of this game-theoretic model to real-world interactions, such as litigation, tax evasion, and domestic or international diplomacy. We also suggest modifications of the game either for use in class or as student exercises. For reference, we conclude with a brief history of game-theoretic treatments of poker.

1. Introduction

We believe that, to be effective, classroom games should meet three criteria. First, they should be simple enough to be readily understood by students. Second, they should be engaging enough to capture students' attention and interest. Third, they should be rich enough not only to describe a particular bit of theory, but also to illustrate real-world phenomena. Our classroom game, Stripped-Down Poker, satisfies these three criteria and has served as a memorable learning experience to students in our undergraduate classes on game theory. From one simple game, students obtain insights about signaling games, bluffing, converting from extensive to strategic form, mixed-strategy equilibrium, Bayes' Theorem, and the value of private information. The game is especially timely given the recent surge of interest in poker in popular culture: we find that a growing number of our students play poker or watch it on television.

2. Playing the Game

We like to play Stripped-Down Poker as a game between the instructor and a student volunteer. To introduce the game to your class, emphasize that you will be playing a simplified version of poker for real money, and that the volunteer will be responsible for paying any losses out of his or her pocket.[2] One way to heighten the interest level of the students is for you to pull out money from your own wallet.

Next, describe the nature of the game. Produce the special deck for the game, explain that it contains four kings and four queens, and choose a student from the front row to verify the deck's contents to the rest of the class. (This student might also serve as the "dealer" when you actually play the game.) Explain that only the professor receives a card during the game; the student player does not. After each player puts an ante of one dollar into the pot, you (the professor) will receive a randomly drawn card, privately observe what it is, and decide whether to fold or to bet. If you fold, the game ends and the student wins the pot, gaining your dollar ante. When betting, you place another dollar in the pot, after which the student has the option to either fold or call. A fold by the student ends the game, this time with you winning the pot and gaining the student's dollar ante. Calling requires the student to add another dollar to the pot, and you to reveal your card. You win the pot with a king, and lose the pot with a queen. The pot at this stage is four dollars, so the winner nets two dollars from the loser.[3]

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[2] Mark hands the volunteer $5 and says that he plans to try to win as much of it back as he can. David imposes a mandatory lab fee of $30 per student at the beginning of his game-theory course, and uses the cash to pay earnings to students for various games they play over the course of the semester. When he earns money from students in a game like Stripped-Down Poker, he adds those earnings to the pool for payoffs in games later in the semester. In nine semesters of teaching this way, no student has yet complained about the lab fee, and indeed the students really seem to enjoy playing games for real money. A nice side effect of this policy is that it generally transfers money from those students who don't attend class regularly to those students who do.

[3] Gardner (2003) uses aces versus kings for his version of our game, while Thomas (1984) uses aces versus twos. (See Section 6 for more on these presentations.) The choice of card types is arbitrary, but we note that using kings versus queens avoids any confusion in the students' minds about whether aces are high or low.

We also like a third option, using a larger deck with equal numbers of face cards versus numbered cards, with four jacks, four queens, and four kings all equally "high" cards, and four twos, four threes, and four fours all equally "low" cards. The advantages are that a larger deck is easier to shuffle, and can easily be modified by adding four fives and four sixes to make the game fair: we show below that in equilibrium the players' expected payoffs are equal when the ratio of low to high cards is 5:3. The disadvantage is the difficulty of making sure the students understand that different face cards are really equivalent.

After describing the rules, ask for a student volunteer.[4] Repeat the game 10-20 times. After a number of rounds, you might ask for a new volunteer to play against you, either because the first volunteer doesn't want to play anymore, or because you want to give several different students the opportunity to play.   We find that students get actively engaged in the game even when they are not playing; they watch attentively, shout out advice to the student player, and generally exhibit high energy levels.[5]

At first glance, many students believe this to be a fair game. This naïve belief serves two important purposes. First, it produces more willing volunteers. Second, it makes the game especially effective for teaching purposes: over the course of repeated play, the students begin to observe that the professor is winning money on average from the student volunteers. This strongly motivates the students for the subsequent analysis of the game.

For maximum effect, you should attempt to play according to the equilibrium (minimax) strategy, which is to bet always when you have a king and to bet one-third of the time when you have a queen. We find it best to use an external randomization device, such as the second hand of one's watch. For example, if the number of seconds in the minute is between 0 and 20, you would bet with a queen, but if it is between 20 and 60, you would fold with a queen. It's crucial that you look at your watch in the same way even if you've drawn a king; otherwise, glancing at one's watch could serve as a "tell" to the students that you are holding a queen, and thus destroy your bluffing advantage.[6] When executing this strategy correctly, the expected payoff turns out to be $1/3 to the professor (-$1/3 to the student), no matter what action the student chooses. After 10-20 repeated rounds of the game, we recommend beginning the discussion by asking the students whether the game is fair or not. Some students may still think the game is fair, and that you were just lucky to have won a positive amount from the students after 20 rounds. Others, after seeing the professor's profits, may now believe the game is unfair, but be uncertain where the unfairness comes from. As a cliffhanger, we often choose to schedule this game for the last twenty minutes of one class period, and challenge the students at the end of class to see if they can figure out whether it is a fair game before the next class period, when we do a thorough analysis.

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[4] Mark likes to describe the game on the chalkboard as if it were just another lecture, then dramatically pull out a deck of cards and his wallet and ask, "Who wants to play?"

[5] For those interested in greater class participation, we note that your students can now play the game online using Charlie Holt's Vecon Lab site <http://veconlab.econ.virginia.edu/sg/sg.htm>, whose signaling game now includes a pre-parameterized Stripped-Down Poker option. On this site, all the students can play against each other, with random assignment of roles into "professor" or "student."   One could schedule the game to be played during class time in a computer lab, or from students' home computers during some time outside of class.

[6] The first time David tried this method to randomize, one of his more observant students informed him after class that he had looked harder at his watch after drawing a queen than he had after drawing a king, thus revealing what card he had. Fortunately, not all students noticed this "tell," but the incident points out how difficult it can be to adopt an appropriate "poker face" for this game.

3. Analytical Exposition

There are a number of directions that you can take the classroom discussion. What you decide to focus on will depend on the level of your course and the ideas that you most wish to illustrate using the game. The following subsections focus on separate potential topics. First, we help the students develop an intuitive understanding of the value of bluffing. Second, we show students how to represent the game in both extensive and strategic form. Third, we solve for the mixed-strategy Nash equilibrium to the game, with probabilistic bluffing and calling. Fourth, we illustrate the use of Bayes' Rule by considering the probability of the professor having a king given that he has bet. Finally, we use Stripped-Down Poker as a very simple signaling game to illustrate signaling equilibrium concepts. These subsections appear in a logical progression, but an effort has been made to make them modular. Feel free to select the combination of topics that most appropriately meets the needs of your class.

3.1 How Should the Players Play?

We prefer to start our discussion of the game by asking the students some questions to get them thinking and to gauge their understanding. First, you might take a quick survey to see how many students believe that the game is a fair one. You might ask one student to explain why he thinks the game is fair, and ask another student why she thinks it is unfair. Then promise the students a definitive answer in the analysis to come.

Second, you can ask the students what they would do if they were playing in the professor's position. Many students believe that the professor's optimal strategy would be to always bet on kings and always fold on queens. When a student suggests this strategy, ask the class what the student player's best response would be. With a little thought, they should be able to see that if the professor always folds with queens, then the only time the student gets to move is when the professor must have a king, and therefore the student should always fold. Note that with this pair of strategies, the professor earns an expected value of $0. But we've already observed through playing the game that the professor earned quite a bit of money from the students!

Next you can ask what the professor's optimal strategy would be if the student always folded. If there is no chance of the student calling, there is no reason to avoid betting with queens: the professor can bluff the student into folding even though she holds a poor card. The professor's optimal strategy is thus to always bet and earn $1 when the student folds. But if the professor always bets, what is the student's optimal strategy? If the professor always bets, then the student should expect there to be a 50% chance of a king and a 50% chance of a queen if he should choose to call. This means if the student calls, his expected payoff is zero. By contrast, if he folds he is guaranteed to lose $1, so his optimal strategy is to always call.

If the student always calls, then what is the professor's optimal strategy? Then the professor wins by betting with kings but loses by betting with queens, so her optimal strategy is to bet with kings and fold with queens, which takes us back to our starting point. By this intuitive discussion, we manage to demonstrate to students that there is no mutual best response, and hence no pure-strategy Nash equilibrium. This leads us to talk about probabilistic bluffing.

A third question to ask the students is why the professor kept looking at her watch before deciding whether to bet or fold. A few students will likely have noticed this behavior and asked about it during game play. This is a good time to share with the student that you did indeed use your watch as a randomizing device, though we prefer not to say exactly what probabilities we

used until we formally solve the game with the class. By now, though, the students should see intuitively that the professor can't get away with bluffing on queens if she does it all the time, so she might want to consider bluffing randomly with probability less than one.

If your class is not a game theory course you might wish to stop with this intuitive level of analysis and skip ahead to the real-world applications of bluffing described below. For an undergraduate course in game theory, we prefer to continue on with a formal analysis of the game, as described in the next two subsections.

3.2 Extensive and Strategic Form

This game offers a nice example for comparing extensive and strategic forms of the same game. This is a sequential-move game, so one can begin by drawing its extensive form (see Figure 1). You might ask the students how they would draw the game tree, and complete it on the board with their help. We find that most students do not instantly recognize the initial move by "nature" to deal a card to the professor, particularly as we use this game as a very early example of a game of incomplete information. Another common problem is for students is to forget to include an information set indicating that the student has to move without knowing what card the professor holds.
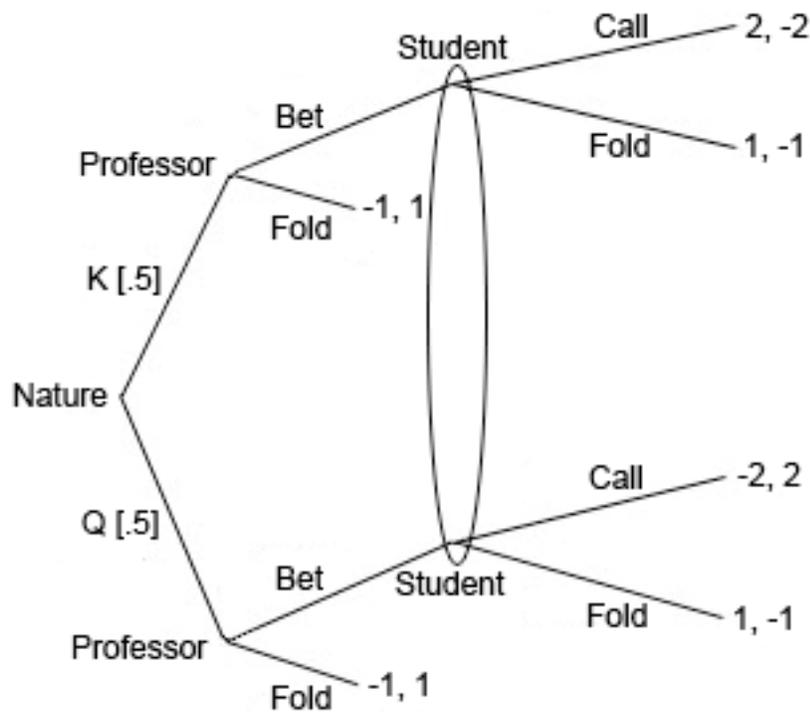

Figure 1. Stripped-Down Poker in extensive form.

Because the student's information set contains two nodes, backwards induction cannot be used to solve the game. This leads us naturally to the idea of converting the game to strategic form and looking for a Bayesian Nash equilibrium using a game matrix.

This is a great opportunity to reinforce a concept that many of our students find difficult. We suggest beginning by asking the students: "How many possible strategies does each player have?" Some students mistakenly believe each player has two strategies, as students easily confuse strategies with actions. Others mistakenly believe that each player has four strategies, as they fail to notice that the student can't condition his strategy on the card type. In fact, the professor has four possible strategies (two actions at each of two decision nodes) and the student has two strategies (two actions at a single information set).

Next, write the 4x2 game matrix up on the board (see Table 1), but allow the students to help you fill in the expected payoffs. Ask the class, for example, "What is the expected payoff to the professor if he always bets and the student always calls?" Give them a minute to think about the question, then ask for a volunteer to explain the answer. Your better students may be able to tell you the answer right away; if not, point out that for this pair of strategies, the possible payoffs are either (-2,+2) when the professor has a king or (+2,-2) when the professor holds a queen. These two cards occur with equal probability, so the expected payoff to each player is zero in the upper-left cell of the payoff matrix.

With that example under their belt, you can ask the students to calculate the rest of the entries in the payoff matrix. To engage the entire class in thought (rather than just the few bright students who always volunteer answers), you might ask the class to take a few minutes to make the calculations at their desks, in consultation with their classmates. You can stroll around the room to monitor their progress, and answer questions that individual students might be more comfortable asking you privately than in front of the entire class. After a few minutes, reconvene the class and ask student volunteers to help you fill out the rest of the payoff matrix on the board. The full matrix will be as shown in Table 1.

Student

	Call	Fold
BB: Bet if K, Bet if Q	0, 0	1, -1
BF: Bet if K, Fold if Q	.5, -.5	0, 0
FB: Fold if K, Bet if Q	-1.5, 1.5	0, 0
FF: Fold if K, Fold if Q	-1, 1	-1, 1

(Professor, rows)

Table 1. Stripped-Down Poker in strategic form.

Next, ask the students whether there are any dominated strategies. Note that the strategies **FB** and **FF** are both strictly dominated for the professor by **BB**, but that **BB** and **BF** are not dominated. After eliminating the strictly dominated strategies (see Table 2), find each player's best responses (see Section 3.1 above) and observe that there is no pure-strategy Nash equilibrium.

	Student	
	Call	**Fold**
BB (Bluff)	0, 0	1, -1
BF (Truthtelling)	.5, -.5	0, 0

Professor (row label, rotated at left)

Table 2. Elimination of dominated strategies, with best responses indicated by underlines.

3.3 Mixed-Strategy Nash Equilibrium

Once pure-strategy Nash equilibria are ruled out, it is time to look for a mixed-strategy equilibrium.

Let **p** represent the probability that the professor plays **BB** (i.e,. that she **Bluffs**).
Let **q** represent the probability that the student plays **Call**.

In order for me to be willing to mix over two strategies, I must be indifferent between them; otherwise, I would play my preferred pure strategy. This must be true for each player, and this insight enables us to find the equilibrium mixing probabilities. To keep each opponent indifferent, we have the following two equations:

$$0\mathbf{p} - .5(1 - \mathbf{p}) = -\mathbf{p} + 0(1 - \mathbf{p}) \implies 1.5\mathbf{p} = .5 \implies \mathbf{p} = 1/3$$

$$0\mathbf{q} + 1(1 - \mathbf{q}) = .5\mathbf{q} + 0(1 - \mathbf{q}) \implies 1 = 1.5\mathbf{q} \implies \mathbf{q} = 2/3$$

Thus the mixed-strategy Nash equilibrium of this game is:

(1/3 **BB +** 2/3 **BF**; 2/3 **Call** + 1/3 **Fold**),

Or, rewriting the professor's strategies with a more intuitive interpretation:

(1/3 **Bluffing +** 2/3 **Truthtelling**; 2/3 **Call** + 1/3 **Fold**).

At this point, you might ask the students to guess what you were doing when looking at your watch during game play. You can then discuss how you used your external randomization device, and take the opportunity to discuss the importance of keeping one's opponent guessing. Ask the students, for example, why you looked at your watch even when you held a king (see Section 2 above).

Next, you can compute the expected value of the game to each player. The professor's expected payoff in equilibrium is a weighted average of the four possible payoffs in Table 2:

$$(2/9)(0) + (1/9)(1) + (4/9)(1/2) + (2/9)(0) = 1/3.$$

Since the game is zero-sum, the student's expected payoff is simply the opposite: -1/3.

We finally have a definitive answer for the students on whether the game is fair or not; the answer is a resounding No. In equilibrium the professor can expect to win one-third of a dollar from the student on each round of the game.[7] The source of the unfairness is information asymmetry: the professor has private information while the student does not, and the professor can use bluffing to press this advantage.

To demonstrate that the advantage comes exclusively from the private information (not, for example, merely from the order of moves) you might ask your students to compare these results to those of a related game with no information asymmetry. There are two obvious ways to do this: either deal the card face up so that both players see it, or deal it so that neither player sees it. In the first case ("Face-Up Stripped-Down Poker"), the student should clearly call the bet only if the card is a queen, and given that strategy, the professor should clearly bet only if the card is a king. Thus, in equilibrium, the professor wins a dollar if the card is a king and loses a dollar if the card is a queen, so the expected payoff is zero. In the second case ("Blind Stripped-Down Poker"), the professor must bet before knowing the value of the card. In this case, the professor has a single information set when she moves, so the strategic form is a 2x2 table instead of a 4x2 table. This game has a pure-strategy equilibrium where the professor always bets, the student always calls, and the equilibrium payoff to each player is again zero. Thus, when both players have symmetric information, it is a fair game after all.

3.4 Bayes' Rule

Stripped-Down Poker is ideal for illustrating the use of Bayes' Rule to update probabilities. After discussing the fact that you actually played your equilibrium mixed strategy during the game, always betting with kings but betting only one-third of the time with queens, ask the students the following question: "Given that I have bet, what is the probability that I'm holding a king?" Typical student responses are one-half or two-thirds, neither of which is correct. Point out that these beliefs crucially determine the student's optimal decision whether to bet or fold. If the student underestimates the probability that the card is really a king given that the professor has bet, he may call more often than is optimal, which the professor could then exploit (see footnote 7).

Now the students should be motivated to learn Bayes' Rule. The rule can be illustrated quite simply for this game using a square diagram (see Figure 2). The left half of the square represents the state of the world in which the professor gets a king (half the area of the square; probability is 1/2); the right half represents the state in which she gets a queen (the remaining half of the area and probability). The square is divided vertically into thirds to represent the possible mixing realizations: in the left half the professor bets in each third; in the right half she bets in only one of the thirds.

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[7] Since the game is zero-sum, each player's strategy in a Nash equilibrium is a minimax strategy: each player can unilaterally guarantee herself the equilibrium payoff by playing her minimax mixture, no matter what non-equilibrium strategy the other player may choose. If the student, for example, were to call more than two-thirds of the time, thus deviating from the equilibrium, the professor will still win on average 1/3 dollar on each play of the game if she plays her strategy of bluffing 1/3 of the time. Of course she would do even better by bluffing less often: in the extreme case where the student always calls, the professor could earn a profit of $0.50 per game (instead of $0.33) by *never* bluffing – never betting with a queen. However, the student would likely begin to notice that the professor bets only with a king and begin folding whenever she bets.

	K	Q	
0-20 Sec	Bet	Bet	1/3
20-40 Sec	Bet	Fold	1/3
40-60 Sec	Bet	Fold	1/3
	1/2	1/2	

Figure 2. A Geometric Exposition of Bayes' Rule.

The diagram's six segments each represent a probability of 1/6. In four of the segments (probability = 4/6 = 2/3) the professor bets, and in three of those she has a king – i.e., 3/4 of her bets are made with a king and 1/4 with a queen. This visual, geometric intuition helps students understand conditional probabilities and Bayes' Rule, concepts they generally find confusing and unintuitive.

With the diagram in hand, the corresponding algebra for the probability that the professor bets, and for the conditional probability that she has a king, is easier for students to follow: let $P(K)$ and $P(Q)$ represent the probabilities of being dealt a king and a queen, respectively, and let $P(B)$ be the probability that the professor bets. Using standard notation to denote conditional probabilities, we first obtain $P(B)$,

$$P(B) = P(B|K)P(K) + P(B|Q)P(Q)$$
$$= (1)(1/2) + (1/3)(1/2)$$
$$= 1/2 + 1/6$$
$$= 2/3,$$

and then we use Bayes' Rule to obtain $P(K|B)$,

$$P(K|B) = P(K \text{ and } B) / P(B)$$
$$= P(B|K)P(K) / [P(B|K)P(K) + P(B|Q)P(Q)]$$
$$= (1)(1/2) / [(1)(1/2) + (1/3)(1/2)]$$
$$= (1/2) / (2/3)$$
$$= 3/4$$

So, given that the professor has bet, the probability of a king is not 1/2, not 2/3, but actually 3/4. With this knowledge, we can check to make sure that the student actually does want to play his proposed equilibrium strategy.[8] If the student folds, his payoff is -$1. If he calls given that the professor bets, he has a probability of 3/4 of losing $2 and a probability of 1/4 of winning $2, for an expected payoff of -$1. So he is indeed indifferent between calling and folding, and thus should be willing to follow his equilibrium mixture.

3.5 Poker as a Signaling Game

Since real-world poker clearly involves strategic signaling and screening, we like to use Stripped-Down Poker as a simple introduction to the topic of dynamic games of incomplete information. Most games designed to illustrate signaling equilibria, such as the well-known "beer/quiche game" (Cho and Kreps, 1987) or the Attacker/Defender game used in Dixit and Skeath (2004, Section 9.5), are slightly more complicated. They involve a first mover with two types and two possible actions, and a second mover who has two possible actions no matter what the first mover does. Our game simplifies signaling even further by eliminating the second mover's ability to act if the first mover chooses to fold. This gives us a 4x2 strategic-form game matrix, in contrast to the 4x4 strategic form of these other standard signaling examples. Though this eliminates some of the richness of a full signaling model, it also yields a big pedagogical advantage. We find that students understand games like beer/quiche much better after they have been introduced to fundamental signaling concepts in Stripped-Down Poker.

Game-theory textbooks introduce students to a taxonomy of three types of signaling equilibria: separating equilibria, pooling equilibria, and semi-separating equilibria. Students usually do not find these concepts to be terribly intuitive, in part because the standard signaling games are conceptually difficult and complicated to solve in strategic form. Semi-separating equilibria are particularly subtle and difficult to follow, as they involve only a partial separation of types. We find that most students don't really even grasp the meaning of "partial separation of types" without the benefit of a simple example, and Stripped-Down Poker provides a very good one.

Why does Stripped-Down Poker have a semi-separating equilibrium? Recall that a pooling equilibrium involves both types taking the same action. In Stripped-Down Poker this would mean that the professor always bets, with both kings and queens,[9] so that the student could not tell what card she holds. By contrast, a separating equilibrium involves each type taking a separate action. In Stripped-Down Poker this would mean that the professor bets with kings but folds with queens,[10] so that the student could infer from a bet that the card was a king. In subsection 3.1 above, we ruled out both the separating strategy and the pooling strategy, showing that neither could be part of an equilibrium. The equilibrium is in fact semi-separating, because

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[8] By doing so, we are demonstrating that what we have found is a Perfect Bayesian equilibrium. That is, we are showing that in addition to being a Nash equilibrium in expected payoffs, the strategies satisfy the refinement that they remain optimal for the players given that their beliefs are updated via Bayes' Rule. If your course includes the topic of Perfect Bayesian equilibrium, you may wish to point this out as a simple example of the concept, though the refinement has no "bite" in this game.

[9] An alternative pooling equilibrium might have the professor fold with both kings and queens, but we've seen that this is a strictly dominated strategy so it cannot be part of an equilibrium.

[10] An alternative separating equilibrium might have the professor fold with kings and bet with queens, allowing the student to infer a queen when he sees a bet. This is clearly a bad strategy, and we have already seen that it's strictly dominated, so it is not part of an equilibrium.

observing a bet by the professor does give some information about the card type, but it does not completely identify the card type. In particular, observing the professor to bet allows the student to update the probability of a king from 1/2 to 3/4, but not all the way to 1. Semi-separating equilibria typically involve mixed strategies.[11]

One difference between Stripped-Down Poker and standard signaling models is that in the latter, one type of first mover typically wants to signal his type truthfully, while one type does not. For example, in the beer/quiche game the strong type wants the would-be bully to know that he really is strong, but the weak type wants the bully to think that he's really a strong type. By contrast, in Stripped-Down Poker the professor never wants the student to know her type truthfully. If she has a king, she would like the student to think she has a queen, so that the student will call and she can win $2 instead of $1. If she has a queen, she would like the student to think she has a king, so that the student will fold and she can win $1 instead of losing $2. When discussing signaling models, it may be worth pointing out this difference.

Real-world versions of poker, such as Texas Hold'em, have some of this flavor. A player with a weak hand may sometimes wish to bluff in order to get others to fold, and a player with a strong hand may sometimes wish to "slow play" in order to keep others in the game and win more money from them. But real-world poker is also much richer and more complicated, with multiple rounds of betting and multiple rounds of cards being dealt. Sometimes real poker players do want to signal truthfully. For instance, a player may think he likely has the current best hand with three of a kind, but worries that if others stay in the betting one of them may eventually draw a card that completes a flush and thus beat him. By betting aggressively, the player can drive out competition and win the hand before such a reversal of fortune. Poker may also involve screening, as when an early player deliberately bets in order to see how a later player will respond; if the later player raises my bet, for example, I may conclude that he likely has a very strong hand and I'll therefore fold.

In a small class where many students are quite familiar with more complicated versions of poker, real-world poker examples might lead to useful discussions. Instructors who have time and interest may wish to refer to some of the wealth of anecdotal examples found in popular books on poker.[12] In order to avoid turning off students not interested in Texas Hold'em, we usually prefer to keep such discussion to a minimum and instead move on to the applications in the next section.

4. Applications

Stripped-Down Poker is an extremely simple two-player game, yet it can roughly model intriguing real-life interactions. Stripped-Down Poker is isomorphic to any signaling-type game where one of the signaler's actions could be thought of as ending the game. This terminal action

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[11] In Problem 2 in the Appendix there is a pure-strategy equilibrium that could plausibly be called semi-separating. There are three types (jack, queen, king) but only two possible actions, so even though the equilibrium involves a pure strategy for the first player, the second player only gets partial separation of types.

[12] Herbert O. Yardley's classic *The Education of a Poker Player* (1957) is both a poker manual and collection of colorful poker stories. Doyle Brunson's 600-page book *Super System* (1978) is something of a bible on poker strategy. Countless books have been written since. Sklansky's *The Theory of Poker* (1987) is the most technical treatment of real-world poker strategy, with plenty of references to game theory and a "Fundamental Theorem of Poker." Most good poker texts emphasize the importance of a poker player's ability to read and adapt to other players, implying that many game-theoretic principles are of more limited use at the table. This sentiment has been echoed in the online poker community by Caro (2003).

can be thought of as a fold, while the other action(s) would represent a bet. The first two examples below are fleshed out in the Appendix with numerical exercises suitable for student problem sets; both have semi-separating equilibria just like those of the classroom game.

Litigation is one very nice example.[13] A plaintiff sues for damages in civil court. The defendant has private information about whether or not he is really guilty, but the plaintiff only has an assessment of the probability of guilt. For simplicity, assume that should the case go to trial, then the court will find out the truth, finding for the plaintiff if and only if the defendant is guilty. Before going to trial, the defendant has the option of offering a settlement to the plaintiff. If he offers a generous settlement, the plaintiff will accept and drop the case. As the game would end here, this action may be thought of as a "fold" by the defendant. On the other hand, the defendant could "bet" by offering a stingy settlement, to which the plaintiff may respond by either accepting the stingy settlement (folding) or rejecting it (calling). If losing in court is costly to the plaintiff, then straightforward play by the defendant (making a stingy offer if and *only* if he's innocent) would induce the plaintiff to always accept the stingy offer. But the defendant will sometimes make the stingy offer even when he's guilty (a bluff), and therefore the plaintiff must sometimes go to court in response to the stingy offer, even though it seems to signal a strong hand for the defendant (i.e., that he's innocent).

A similar example involves tax evasion and auditing.[14] Consider the reporting of income from tips by a waitress to the Internal Revenue Service. Both parties may have the same priors about the probability of earning high versus low tip income, but only the waitress knows how much was actually realized in tips. The waitress may report the truth and pay the appropriate taxes (a fold), or she may underreport high tip income and pay less in taxes (a bet). When the IRS receives a high tax payment, it does not suspect any problem and takes no action (i.e., the game ends). When it receives a low tax payment, it either chooses to audit the waitress (call) or to accept the waitress's reporting (fold). If found guilty of underreporting her tips, the waitress will be responsible for her true tax liability and perhaps an additional penalty as well. Both parties incur costs in the case of an audit, whether the waitress is found innocent or not. As both of these examples show, the game need not be zero-sum to benefit from the insights of Stripped-Down Poker; interesting efficiency issues ("What are the social costs of equilibrium auditing?") can arise in these games.

Other applications relate to politics at all levels, from international to domestic (household) conflicts. A dictator suspected of harboring weapons of mass destruction may comply in full with international inspections (a fold), or may obstruct such inspections in order to maintain his threat of power at the risk of military intervention (a bet). A schoolboy required to complete homework before playing with friends may actually do his homework first (a fold) or may simply tell his mother that it is done (a bet), possibly gaining more time to play at the risk of being caught.

These applications also present an opportunity to discuss with the students the question of choosing an appropriate economic model. Depending on one's willingness to simplify each of the above situations, one might model the actions available to the first mover as binary, multiple yet discrete, or continuous. The binary case fits the poker model exactly; the agent literally has only two options available, which are then precisely equivalent to a fold and a bet in Stripped-Down Poker. However, real life situations very often offer a range of intermediate alternatives, sometimes even a continuum of them. The waitress may report an intermediate level of tip

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[13] See Exercise 3 in the Appendix.
[14] See Exercise 4 in the Appendix.

income. The schoolboy may do part of his homework or do it sloppily. The dictator may come clean about certain weapons programs but remain ambiguous about the status of others. You might encourage students to extend this model to make it more realistic: when the number of actions available to the first mover is more than two but still discrete and relatively small, Stripped-Down Poker may be extended to include these possibilities; we consider this possibility in the next section.

5. Extensions and Exercises

One of the greatest virtues of Stripped-Down Poker is its simplicity. However, if more complex examples are desired, the game lends itself very well to a host of extensions to challenge your students.

Perhaps most interesting, one might ask, "How would we have to change the composition of the deck in order to make this a fair game?" We find this to be a challenging extra-credit problem for our students to consider. The answer turns out to be a deck with 3/8 kings and 5/8 queens, so the "fair" game can be conveniently played by replacing one of the four kings in the original deck with a fifth queen. A sketch of the solution to this problem can be found in Exercise 1 of the Appendix.

Second, one might add other types of cards to the deck. For example, a version of Stripped-Down Poker could be played where the deck contains kings, queens, and jacks. The first mover draws a card, privately views it, and either bets or folds. The second mover may then fold or call as before. The king wins for the professor, the jack gives the win to the student, and the queen splits the pot between the two players. This game turns out to have an 8x2 strategic-form game table with one pure-strategy equilibrium,[15] and the game is tractable enough to be an excellent out-of-class exercise to test the students' understanding of converting imperfect-information games from extensive to strategic form.[16] Supplementing the deck with additional card types allows Stripped-Down Poker to model a limited number of multiple discrete first-mover actions, which is the extension alluded to at the end of section 4.

Third, one might change the information dynamics by having each player draw a card. For a player to win the pot in a showdown, she must hold a card higher in value than her opponent; if both players hold cards of the same value, there will be a split pot. This eliminates the first player's information advantage, and thereby creates a fair game. To understand this point about removing the information advantage, students may find it enlightening to solve for the two pure-strategy Nash equilibria to this game, each of which has an expected payoff of zero to each player. There are a number of possible variations to consider; we recommend exercise 9.8 of Dixit and Skeath (2004) as one interesting and tractable example.[17] It is also possible to think about extending the game to have multiple rounds of betting, more than two players, and/or more than two card types, in order to address questions of first-mover or last-mover advantage in poker and other information games.[18] These extensions may, however, be beyond the reach of all but the best undergraduates.

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[15] The game also has an infinite number of mixed-strategy equilibria.

[16] See Exercise 2 in the Appendix.

[17] This exercise is very similar to von Neumann's rather than Borel's poker model (see section 6). In addition, it has the feature that the amount of the bet is twice the amount of the initial ante. This model proves to be a bit more interesting (with a unique mixed-strategy equilibrium) than the two-card version of Borel's model, which has a somewhat uninteresting pure-strategy equilibrium.

[18] See, for example, Karlin and Restrepo (1957).

6. Historical Treatments of Poker in Game Theory

There is nothing new about game-theoretic models of poker. John von Neumann, the father of game theory, formulated two models that served to illustrate the rationality of bluffing.[19] But even he was preceded in this, by French mathematician Émile Borel.[20,21] For simplicity, both Borel's and von Neumann's models feature risk-neutral players and a continuum of possible hands.[22, 23]

Borel's poker model works as follows. After each player antes one dollar into the pot, each draws a hand represented by a draw from a uniform distribution on [0,1]. After the players privately observe their own hands, the first player moves by either folding or raising. If the first player folds, he forfeits his ante to the second player and ends the game. If the first player raises, he adds an additional dollar to the pot. Then the second player moves by either folding or calling. If the second player folds, she forfeits her ante to the first player and ends the game. If she calls the first player's bet, she adds an additional dollar to the pot, and both players reveal their hands in a "showdown." At this point, the player with the highest hand wins the pot, for a net gain of two dollars (the ante plus the additional bet).

Von Neumann's second poker model is similar to Borel's, the only difference being that the first player's option to fold is replaced with the option to "check" – that is, to stay in the game without raising the stakes. When the first player checks, the second player does not have an opportunity to move, and the holder of the higher card wins the other player's ante.

Borel's model yields equilibrium bluffing, in the sense that sometimes a player bets despite knowing that she may have the weaker hand, and her opponent sometimes responds by folding when full information would have revealed that he had the stronger hand. Von Neumann's model results in even stronger bluffing: the first player bets in equilibrium even with the worst possible hand. Though interesting for very advanced analytical courses, the models of Borel and von Neumann would have limited pedagogical value in most undergraduate courses. First, students find it much more engaging to play a poker game with a real deck of cards rather than a continuous distribution of hands. Second, these models are complicated to analyze: Borel's model would probably be grasped only by mathematically mature undergraduates, while von Neumann's model is probably best left for graduate courses.

Since the time of von Neumann, researchers have analyzed many other models of poker and other bluffing games, including the analytical models of Bellman and Blackwell (1949), Kuhn (1950), Nash and Shapley (1950), Isaacs (1955), Karlin and Restrepo (1957), Goldman and Stone (1960), and Friedman (1971). The latter paper analyzed an abstract game, framed in terms of actual poker, of which our game is a special case. Burns (2005) developed a suite of four simplified poker games to study judgment and decision making in the laboratory, but even these are too complicated for in-class analytical solution. Of these, Kuhn's poker model is the most

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[19] von Neumann and Morgenstern (1944).

[20] Borel (1938).

[21] For a detailed analysis of the poker models of Borel and von Neumann, see Ferguson and Ferguson (2003).

[22] While a poker game with an infinite number of potential hands may seem far removed from reality, the number of distinct 5-card poker hands that can be drawn from a standard 52-card deck is larger than one might naively guess: the number of possible combinations of 52 cards taken 5 at a time yields 2,598,960 distinct hands.

[23] Actually, Von Neumann's first poker model featured a discrete number of possible hands $s_i \in \{1, 2, \ldots, S-1, S\}$. In all other respects, it was identical to the second poker model.

interesting for pedagogical purposes: it is essentially von Neumann's model with exactly three possible hands.

Stripped-Down Poker is an even simpler variation on Borel's model, in which only the first player gets a card and there are only two card types in the deck. We were inspired to develop this classroom game by an example in the textbook by Roy Gardner (2003), who calls his version of the game "Liar's Poker." L. C. Thomas (1984) gave a similar presentation in his textbook, calling the game "Simplified Poker." The only difference in rules between these two games and our classroom game is that the other authors require the first mover to bet when the high card is drawn (instead of giving him the option to bet or fold), an assumption that yields the same equilibrium behavior. Rapoport et al. (1997) conducted laboratory experiments with the game; their game differs from ours only in the bet size and thus the bluffing and calling mixtures. Our contribution has been to transform the model from a simple textbook exercise into an engaging classroom game, vividly and concretely illustrating multiple learning objectives. We believe this is the simplest possible version of poker, and as discussed above, also the simplest possible example of a signaling game.

7. Conclusion

We have presented a simple classroom game based on poker that provides significant instructional benefits with a minimum of materials and preparation. Stripped-Down Poker is thoroughly engaging in a classroom setting, and its analysis is accessible to students. The game lends itself to a number of interesting variations and applications which can be explored by students both in class and in out-of-class exercises. We find it a useful, effective, flexible, and memorable teaching tool. And it's fun!

References

Bellman, Richard, and David Blackwell. "Some Two-Person Games Involving Bluffing." *Proceedings of the National Academy of Science of the United States of America*: vol. 35, no. 10, 1949, pp. 600-605.

Borel, Émile. *Traité de Calcul des Probabilités et ses Applications*, vol. 4, Gauthier-Villars, 1938.

Brunson, D. *Super System, 3rd edition*. New York: Cardoza Publishing, 1978.

Burns, Kevin. "Pared-down Poker." Working paper, The MITRE Corporation, 2004.

Caro, Mike. "Game Theory Simplified, And Why Fixed-Strategies Fail." Poker Articles, Pokerpages.com: <http://www.pokerpages.com/articles/archives/caro21.htm>, May 8, 2003.

Cho, In-Koo and David M. Kreps. "Signaling Games and Stable Equilibria." *Quarterly Journal of Economics*, vol. 102, 1987, pp. 179-222 (see pp. 183-187).

Dixit, Avinash, and Susan Skeath. *Games of Strategy*. New York: W.W. Norton and Company, 2004.

Ferguson, Chris, and Thomas S. Ferguson. "On the Borel and von Neumann Poker Models." Working paper, University of California at Los Angeles, 2004.

Friedman, Lawrence. "Optimal Bluffing Strategies in Poker." *Management Science*: vol. 17, no. 12, 1971, pp. 764-771.

Gardner, Roy. *Games for Business and Economics*. Hoboken, New Jersey: John Wiley & Sons, 2003.

Goldman, A. J. and J. J. Stone. "A Symmetric Continuous Poker Model." *Journal of Research of the National Institute of Standards and Technology*: vol. 64B, 1960, pp. 35-40.

Karlin, S. and R. Restrepo. "Multi-Stage Poker Models." *Contributions to the Theory of Games, III*. (H.W. Kuhn and A.W. Tucker, eds.?) Annals of Mathematics Studies, Number 39. Princeton, New Jersey: Princeton University Press, 1957, pp. 337-363.

Kuhn, H. W. "A Simplified Two-Person Poker." *Contributions to the Theory of Games, I*. H. W. Kuhn and A. W. Tucker, eds. Annals of Mathematics Studies, Number 24. Princeton, New Jersey: Princeton University Press, 1950, pp. 97-103.

Isaacs, R. "A Card Game with Bluffing." *American Mathematical Monthly*: vol. 62, 1955, pp. 99-108.

Nash, J. F. and L. S. Shapely. "A Simple Three-Person Poker Game." *Contributions to the Theory of Games, I*. H. W. Kuhn and A. W. Tucker, eds. Annals of Mathematics Studies, Number 24. Princeton, New Jersey: Princeton University Press, 1950, pp. 105-116.

Rapoport, Amnon, Ido Erev, Elizabeth V. Abraham, and David E. Olson. "Randomization and Adaptive Learning in a Simplified Poker Game." *Organizational Behavior and Human Decision Processes*: vol. 69, 1997, pp. 31-49.

Sklansky, David. *The Theory of Poker*. Las Vegas: Two Plus Two Publishing, 1989.

Thomas, L. C. *Games, Theory and Applications*. New York: Ellis Horwood Limited, 1984.

von Neumann, John and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton, New Jersey: Princeton University Press, 1944.

Yardley, H. O. *The Education of a Poker Player*. New York: Simon and Schuster, 1957.

Appendix: Suggested Exercises.

1. What mix of kings and queens is required to make Stripped-Down Poker a fair game (i.e., such that the value to both players is zero)?

    [To solve this problem, first find the payoffs in the normal-form game in which k is the fraction of kings in the deck. Note that the game is not fair either when $k = 0$ or when $k = 1$. Also, note that when $k > 0$ the strategies **FB** and **FF** are dominated for the professor by **BB** and **BF** respectively. Next, find the mixed strategy equilibrium of the remaining two-by-two normal-form game. The equilibrium mixture for each player will be a function of k instead of a fixed value. Now find the expected value of the game for the professor by multiplying the expected payoff of each cell of the game table by the probability that that cell is reached in equilibrium and then summing the four resulting products. Set this expression equal to zero and solve for k. The answer turns out to be $k = 3/8$.]

2. In class, we played a simplified version of poker, played by the professor against a student. We each ante, then I draw a card, then I bet or fold, and if I bet, you either call or fold. If I fold, you win my $1 ante. If you fold, I win your $1 ante. If I bet and you call, I win $2 from you ($1 ante plus $1 additional bet) if I have a king, and lose $2 to you if I have a queen. We computed the equilibrium to that game in class.

    Now let's consider a slightly more complicated version of that game. Suppose that I now use a deck with three types of cards: 4 kings, 4 queens, and 4 jacks. All rules remain the same as before, except for what happens when I bet and you call. When I bet and you call, I win $2 from you if I have a king, we "tie" and each get back our money if I have a queen, and I lose $2 to you if I have a jack.

    a. Draw the game tree for this game. Label the two players as "professor" and "student." Be careful to label information sets correctly. Indicate the payoffs at each terminal node.

    b. How many pure strategies does the professor have in this game? Explain your reasoning.

    c. How many pure strategies does the student have in this game? Explain your reasoning.

    d. Represent this game in strategic form. This should be a matrix of *expected* payoffs for each player, given a pair of strategies.

    e. Find the unique pure-strategy Nash equilibrium to this game. Explain in English what the strategies are.

    f. Would you call this a pooling equilibrium, a separating equilibrium, or a semi-separating equilibrium?

    g. In equilibrium, what is the expected payoff to the professor of playing this game?

3. Corporate lawsuits may sometimes be signaling games. Here is one example. In 2003, AT&T filed suit against eBay, alleging that its Billpoint and PayPal electronic-payment systems infringe on AT&T's 1994 patent on "mediation of transactions by a communications system." Let's consider this situation from the point in time when the suit was filed. In response to this suit, as in most patent-infringement suits, eBay can offer to settle with AT&T without going to court. If AT&T accepts eBay's settlement offer, there will be no trial. If AT&T rejects eBay's settlement offer, the outcome will be determined by the court.

The amount of damages claimed by AT&T is not publicly available. Let's assume that AT&T is suing for $300 million. In addition, let's assume that if the case goes to trial, the two parties will incur court costs (paying lawyers and consultants) of $10 million each.

Because eBay is actually in the business of processing electronic payments, we might think that eBay knows more than AT&T does about its probability of winning the trial. For simplicity, let's assume that eBay knows for sure whether it will be found innocent (i) or guilty (g) of patent infringement. From AT&T's point of view, there is a 25% chance that eBay is guilty (g) and a 75% chance that eBay is innocent (i).

Let's also suppose that eBay has two possible actions: a generous settlement offer (G) of $200 million or a stingy settlement offer (S) of $20 million. If eBay offers a generous settlement, assume that AT&T will accept, thus avoiding a costly trial. If eBay offers a stingy settlement, then AT&T must decide whether to accept (A) and avoid a trial, or reject and take the case to court (C). In the trial, if eBay is guilty it must pay AT&T $300 million in addition to paying all the court costs. If eBay is found innocent it will pay AT&T nothing, and AT&T will pay all the court costs.

a. Write down the extensive-form game tree for this game. Be careful about information sets.

b. Which of the two players has an incentive to bluff in this game? What would bluffing consist of? Explain your reasoning.

c. Write down the strategic-form game matrix for this game. Find all of the Nash equilibria to this game. What are the expected payoffs to each player in equilibrium?

4. Wanda works as a waitress, and consequently has the opportunity to earn cash tips that are not reported by her employer to the Internal Revenue Service. Her tip income is rather variable. In a good year (G), she earns high income, so that her tax liability to the IRS is $5000. In a bad year (B), she earns low income, and her tax liability to the IRS is $0. The IRS knows that the probability of her having a good year is 0.6, and the probability of her having a bad year is 0.4, but it doesn't know for sure which outcome has resulted for her this tax year.

In this game, first Wanda decides how much income to report to the IRS. If she reports high income (H), she pays the IRS $5000. If she reports low income (L), she pays the IRS $0. Then the IRS has to decide whether to audit Wanda. If she reports high income, they do not audit, because they automatically know they're already receiving the tax payment Wanda owes. If she reports low income, then the IRS can either audit (A) or not audit (N). When the IRS audits, it costs the IRS $1000 in administrative costs, and also costs Wanda $1000 in the opportunity cost of the time spent gathering bank records and meeting with the auditor. If the IRS audits Wanda in a bad year (B), then she owes nothing to the IRS, although she and the IRS have each incurred the $1000 auditing cost. If the IRS audits Wanda in a good year (G), then she has to pay the $5000 she owes to the IRS, in addition to her and the IRS each incurring the cost of auditing.

a. Suppose that Wanda has a good year (G), but she reports low income (L). Suppose the IRS then audits her (A). What is the total payoff to Wanda, and what is the total payoff to the IRS?

b. Which of the two players has an incentive to bluff in this game? What would bluffing consist of?

c. Write down the extensive-form game tree for this game. Be careful about information sets.

d. How many pure strategies does each player have in this game? Explain your reasoning.

e. Write down the strategic-form game matrix for this game. Find all of the Nash equilibria to this game. Identify whether the equilibria you find are separating, pooling, or semi-separating.

f. Let x equal the probability that Wanda has a good year. In the original version of this problem, we had $x = 0.6$. Find a value of x such that Wanda always reports low income in equilibrium.

g. What is the full range of values of x for which Wanda always reports low income in equilibrium?